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Bicovariant differential calculi on $\mathrm{GL}_{p,q}(2)$ and quantum subgroups

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Abstract. In a recent paper all bicovariant differential calculi on the two-parameter quantum group $\mathrm{GL}_{p,q}(2)$ were determined. In this work we elaborate some of their properties and discuss relations with work by other authors. Furthermore, we show that there are two different ways in which bicovariant differential calculi on $\mathrm{GL}_q(2)$ (where $p = q$) induce corresponding calculi on the quantum subgroup $\mathrm{SL}_q(2)$. It is shown that on $\mathrm{SL}_q(2)$ there are only two different bicovariant differential calculi. From these one obtains the $4D_\pm$ calculi on $\mathrm{SU}_q(2)$. The classical limit of the two differential calculi on $\mathrm{SL}_q(2)$ is investigated revealing a relation with recent work on deformed differential calculi on commutative (function) algebras.

1. Introduction

Groups play an important role in the formulation of physical theories and in solving mathematical problems arising in physics. Quantum groups can be viewed as generalizations of ordinary groups (in the sense of Hopf algebras) and it is of interest to explore their possible relevance in physical contexts.

Differential geometry on Lie groups enters physical theories in various forms. The Lie algebra is represented as the algebra of left- (or right-) invariant vector fields on the group. In field theories with an action invariant under a Lie group these vector fields ('currents') are related to conserved charges via Noether's theorem. In many physical models the (left- or right-invariant) Maurer–Cartan 1-forms on a Lie group play a central role. These notions can be carried over to quantum groups [1]. The quantum analogues of left-invariant vector fields are related to non-local currents and conserved charges in two-dimensional quantum field theories.

Differential calculus on quantum groups enters the formulation of gauge theory with quantum groups [2, 3]. It is central in the approach [4, 5] towards 'q-gauge theory' and 'q-gravity' by 'softening' a quantum group (analogous to the corresponding procedure in the case of group manifolds [6]). Examples of 'quantum spaces' on which quantum groups act are obtained as 'quantum coset spaces' and inherit a differential calculus from that on the corresponding quantum group [7]. Differential calculus is the very basic structure needed to formulate dynamics on 'quantum spaces'.

A general theory of (bicovariant) differential calculus on quantum groups has been developed by Woronowicz [1]. He gave two examples of bicovariant differential calculi on $\mathrm{SU}_q(2)$ which were called the $4D_\pm$ calculi [1] (see also [8–10]). Since then a large number of papers has appeared dealing with examples of bicovariant differential calculi on special quantum groups or with approaches to define 'preferred' bicovariant calculi on certain classes of quantum groups. Formulations of bicovariant calculi on quantum groups

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with an R-matrix were given in [2, 9, 11–14]. A formalism described in [15] was used in [16, 17] to define bicovariant differential calculi on quantized simple Lie groups (see also [5, 18–21]). In [22, 23] special differential calculi on the quantum group $\mathrm{GL}_{p,q}(2)$ were constructed from special differential calculi on a 'quantum plane' on which the quantum group acts (see also [14, 24, 25]). Some of them turned out to be bicovariant. A recipe for the definition of bicovariant differential calculi on more general algebras has been given in [26].

In [27] *all* bicovariant differential calculi on the two-parameter quantum group $\mathrm{GL}_{p,q}(2)$ were found. They form a family which depends on an additional parameter s†. For $s = 0$ one recovers two differential calculi which have already been found [22] and which can be expressed in terms of the R-matrix of the quantum group [13, 14]. The knowledge of *all* bicovariant differential calculi in the example of the quantum group $\mathrm{GL}_{p,q}(2)$ allows us to relate various approaches and to clarify how and to what extent they work. In particular, we obtain some results concerning the relation between bicovariant differential calculi on $\mathrm{GL}_{p,q}(2)$ and calculi on quantum subgroups.

A summary of those results from [27] needed for our discussion is given in section 2 with some improvements and additional material. In [13, 28] questions have been raised concerning the consistency of the differential algebra for $s \neq 0$. This will be addressed in section 3.

In section 4 we discuss how to obtain a differential calculus on the quantum subgroup $\mathrm{SL}_q(2)$ from a differential calculus on $\mathrm{GL}_q(2)$ (where $p = q$) (see also [19, 20]).

Section 5 briefly discusses the quantum subgroups $U_{\bar{q},q}(2)$ and $\mathrm{SU}_q(2)$ and how the $4D_\pm$ calculi on $\mathrm{SU}_q(2)$ [1] are recovered from our results. Section 6 contains some conclusions.

2. Bicovariant differential calculi on $\mathrm{GL}_{p,q}(2)$

The quantum group $\mathrm{GL}_{p,q}(2)$ is the Hopf algebra $\mathcal{A}$ generated by a, b, c, d satisfying the commutation relations

$$\begin{array}{lll} ac = qca & bd = qdb & ad = da + (p - 1/q)bc \\ bc = (q/p)cb & ab = pba & cd = pdc. \end{array} \tag{2.1}$$

and the unit $\mathbb{1}$. *Coproduct* and *counit* are given by‡

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{pmatrix} \qquad \epsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2}$$

and the *antipode* is determined by

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathcal{D}^{-1}\begin{pmatrix} d & -b/q \\ -qc & a \end{pmatrix} = \begin{pmatrix} d & -b/p \\ -pc & a \end{pmatrix}\mathcal{D}^{-1} \tag{2.3}$$

where $\mathcal{D} = ad - pbc = da - q^{-1}bc$. In addition, $\Delta(\mathbb{1}) = \mathbb{1}\otimes\mathbb{1}$, $\epsilon(\mathbb{1}) = 1$ and $S(\mathbb{1}) = \mathbb{1}$.

The central object of (first-order) differential calculus is the *exterior derivative*

$$\mathrm{d} : \mathcal{A} \to \Lambda^1(\mathcal{A}) = \text{space of 1-forms} \tag{2.4}$$

† p, q and s are complex numbers.

‡ We use a compact notation for $\Delta(a) = a\otimes a + b\otimes c$ etc.

satisfying the Leibniz rule

$$\mathrm{d}(fh) = (\mathrm{d}f)h + f\,\mathrm{d}h \qquad \forall f, h \in \mathcal{A}. \tag{2.5}$$

The space of 1-forms $\Lambda^1(\mathcal{A})$ is generated as an $\mathcal{A}$-bimodule by the differentials of a, b, c, d. To specify a (first-order) differential calculus one has to prescribe (consistent) commutation relations between a, b, c, d and their differentials.

A *left-coaction* $\Delta_{\mathcal{L}} : \Lambda^1(\mathcal{A}) \to \mathcal{A} \otimes \Lambda^1(\mathcal{A})$ extends Δ as a bimodule homomorphism to 1-forms such that

$$\Delta_{\mathcal{L}} \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} := \begin{pmatrix} a \otimes \mathrm{d}a + b \otimes \mathrm{d}c & a \otimes \mathrm{d}b + b \otimes \mathrm{d}d \\ c \otimes \mathrm{d}a + d \otimes \mathrm{d}c & c \otimes \mathrm{d}b + d \otimes \mathrm{d}d \end{pmatrix}. \tag{2.6}$$

In the same way a *right-coaction* $\Delta_{\mathcal{R}} : \Lambda^1(\mathcal{A}) \to \Lambda^1(\mathcal{A}) \otimes \mathcal{A}$ is a bimodule homomorphism with

$$\Delta_{\mathcal{R}} \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} = \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.7}$$

The existence of $\Delta_{\mathcal{L}}$ and $\Delta_{\mathcal{R}}$ depends on the commutation relations between a, b, c, d and their differentials†. If both exist, the (first-order) differential calculus is called *bicovariant* [1].

Assuming the existence of $\Delta_{\mathcal{L}}$, there is a basis of (left-coinvariant) *Maurer–Cartan 1-forms* θ^K in $\Lambda^1(\mathcal{A})$ given by

$$\begin{pmatrix} \theta^1 & \theta^2 \\ \theta^3 & \theta^4 \end{pmatrix} = S \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathrm{d} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.8}$$

Commutation relations between the generators of $\mathcal{A}$ and their differentials can be expressed in terms of the Maurer–Cartan 1-forms:

$$\theta^K f = \Theta(f)^K_L \theta^L \qquad \forall f \in \mathcal{A}. \tag{2.9}$$

Compatibility with $\Delta_{\mathcal{L}}$ leads to [1, 27]

$$\Theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.10}$$

where A, B, C, D are 4×4 matrices (with complex entries). Consistency of (2.9) requires $\Theta(fh) = \Theta(f)\Theta(h)$ which means that A, B, C, D have to form a representation of a, b, c, d. (2.9) and (2.10) imply

$$\theta^K a = (aA^K_L + bC^K_L)\theta^L \qquad \theta^K b = (aB^K_L + bD^K_L)\theta^L \tag{2.11}$$

and the corresponding relations with a replaced by c and b replaced by d.

Let us recall the following result from [27].

† In the case of a Lie group left and right coaction reformulate the left and right multiplication on the group. This is explained in detail in [5].

Theorem 2.1. Let $r := pq \neq 0, -1$†. Bicovariant (first-order) differential calculi on $\mathrm{GL}_{p,q}(2)$ are given by‡

$$A = \begin{pmatrix} A^1_1 & 0 & 0 & s \\ 0 & q\beta & 0 & 0 \\ 0 & 0 & p\beta & 0 \\ A^4_1 & 0 & 0 & 1-rs \end{pmatrix} \qquad B = \begin{pmatrix} 0 & B^1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B^3_1 & 0 & 0 & B^3_4 \\ 0 & B^4_2 & 0 & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & 0 & B^1_2 & 0 \\ (q/p)B^3_1 & 0 & 0 & (q/p)B^3_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B^4_2 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1-s & 0 & 0 & D^1_4 \\ 0 & q\beta & 0 & 0 \\ 0 & 0 & p\beta & 0 \\ rs & 0 & 0 & D^4_4 \end{pmatrix} \tag{2.12}$$

where

$$\begin{aligned}
\beta &= (1 + A^1_1 - rs)/(1+r) \\
A^4_1 &= r(1-s) - [2r + (r-1)A^1_1 + r^2(r-1)s]/(1+r) \\
B^1_2 &= -(1 - rA^1_1 + r^2 s)/(1+r) \\
B^3_1 &= p[(1 + A^1_1 + r^2 s)/(1+r) - 1 + s] \\
B^3_4 &= \frac{1}{q}[r(1 + A^1_1 + s)/(1+r) - 1 + s] \\
B^4_2 &= -(r - A^1_1 + rs)/(1+r) \\
D^1_4 &= \frac{1}{r}(1-s) - [2 + (1-r)A^1_1 + r(r-1)s]/(1+r) \\
D^4_4 &= A^1_1 + (1-r)s
\end{aligned} \tag{2.13}$$

and

$$(A^1_1)^2 - \frac{1}{r}[1 + r^2 - (1 + r - r^2 + r^3)s]A^1_1 + 1 - (1 + 2r - r^2)s - (1 + r + r^3)s^2 = 0. \tag{2.14}$$

For $r \neq 1$ these formulae determine *all* bicovariant differential calculi on $\mathrm{GL}_{p,q}(2)$§.

Any set of functions $A^1_1(r)$ and $s(r)$ which solves (2.14) determines a bicovariant differential calculus on the quantum group $\mathrm{GL}_{p,q}(2)$. In terms of the differentials, the commutation relations (2.9) for the bicovariant differential calculi in theorem 2.1 read

$$\mathrm{d}a\, a = (A^1_1 + spr^2\mathcal{D}^{-1}bc)a\,\mathrm{d}a - s\mathcal{D}^{-1}a^2(qc\,\mathrm{d}b + pb\,\mathrm{d}c - a\,\mathrm{d}d)$$
$$\begin{aligned}\mathrm{d}a\, b = {} & p[(r^2 s + A^1_1 + 1)/(1+r) + spr\mathcal{D}^{-1}bc]b\,\mathrm{d}a - [(r^2 s - rA^1_1 + 1)/(1+r) + spr\mathcal{D}^{-1}bc]a\,\mathrm{d}b \\ & - s(p/q)\mathcal{D}^{-1}ab(pb\,\mathrm{d}c - a\,\mathrm{d}d)\end{aligned}$$

† The equations (2.13) only make sense when $r \neq 0, -1$. For some differential calculi the limit towards these values of r may exist, however.

‡ A closer inspection of the exceptional case where $A^1_1 + A^4_4 - (1+r)\beta \neq 0$ which appeared in the analysis in [27, p 1721], shows that the additional assumption of a classical limit made there is unnecessary.

§ For $r = 1$ there are additional calculi [27].

$$
\begin{aligned}
\mathrm{d}a\, c &= q[(r^2s+A^1_1+1)/(1+r)+spr\mathcal{D}^{-1}bc]c\,\mathrm{d}a-[(r^2s-rA^1_1+1)/(1+r)+spr\mathcal{D}^{-1}bc]a\,\mathrm{d}c \\
&\qquad - s(q/p)\mathcal{D}^{-1}ac(qc\,\mathrm{d}b - a\,\mathrm{d}d) \\
\mathrm{d}a\, d &= [1-s(1-pr\mathcal{D}^{-1}bc)]d\,\mathrm{d}a \\
&\qquad - (1/r)[(r^2s-rA^1_1+1)/(1+r)-s(1-pr\mathcal{D}^{-1}bc)](qc\,\mathrm{d}b+pb\,\mathrm{d}c) \\
&\qquad + (1/r)[(r^2s-rA^1_1+1)(1-r)/(1+r)-s(1-pr\mathcal{D}^{-1}bc)]a\,\mathrm{d}d \\
\mathrm{d}b\, a &= [(r^2s+A^1_1-r)/(1+r)+spr\mathcal{D}^{-1}bc]b\,\mathrm{d}a \\
&\qquad - (1/p)[(r^2s-rA^1_1-r)/(1+r)+spr\mathcal{D}^{-1}bc]a\,\mathrm{d}b \\
&\qquad - (s/q)\mathcal{D}^{-1}ab(pb\,\mathrm{d}c-a\,\mathrm{d}d) \\
\mathrm{d}b\, b &= s(p/q)^2\mathcal{D}^{-1}b^2(rd\mathrm{d}a-pb\,\mathrm{d}c+a\,\mathrm{d}d)+[A^1_1-s(r+p\mathcal{D}^{-1}bc)]b\,\mathrm{d}b \\
\mathrm{d}b\, c &= q[(r^2s+A^1_1-r)/(1+r)+s(1+p\mathcal{D}^{-1}bc)]d\,\mathrm{d}a+(q/p)[1-s(1+r+p\mathcal{D}^{-1}bc)]c\,\mathrm{d}b \\
&\qquad -s(1+r+p\mathcal{D}^{-1}bc)b\,\mathrm{d}c+(1/p)[(rs+rA^1_1-1)/(1+r)+s(1+p\mathcal{D}^{-1}bc)]a\,\mathrm{d}d \\
\mathrm{d}b\, d &= sp^2\mathcal{D}^{-1}db(qd\mathrm{d}a-b\,\mathrm{d}c)-q[(rs-A^1_1-1)/(1+r)+(s/q)\mathcal{D}^{-1}bc]d\,\mathrm{d}b \\
&\qquad -[(r^2s-rA^1_1+1)/(1+r)-(s/r)(r+p\mathcal{D}^{-1}bc)]b\,\mathrm{d}d \\
\mathrm{d}c\, a &= -(s/p)\mathcal{D}^{-1}ac(qc\,\mathrm{d}b-a\,\mathrm{d}d)+[(r^2s+A^1_1-r)/(1+r)+srp\mathcal{D}^{-1}bc]c\,\mathrm{d}a \\
&\qquad -(1/q)[(r^2s-rA^1_1-r)/(1+r)+srp\mathcal{D}^{-1}bc]a\,\mathrm{d}c \\
\mathrm{d}c\, b &= p[(r^2s+A^1_1-r)/(1+r)+s(1+p\mathcal{D}^{-1}bc)]d\,\mathrm{d}a \\
&\qquad -s(r+1+p\mathcal{D}^{-1}bc)c\mathrm{d}b+(p/q)[1-s(r+1+p\mathcal{D}^{-1}bc)]b\,\mathrm{d}c \\
&\qquad +(1/q)[(rs+rA^1_1-1)/(1+r)+s(1+p\mathcal{D}^{-1}bc)]a\,\mathrm{d}d \\
\mathrm{d}c\, c &= s(q/p)^2\mathcal{D}^{-1}c^2(rd\,\mathrm{d}a-qc\,\mathrm{d}b+a\,\mathrm{d}d)+[A^1_1-s(r+p\mathcal{D}^{-1}bc)]c\,\mathrm{d}c \\
\mathrm{d}c\, d &= s(q^2/p)\mathcal{D}^{-1}cd(pd\,\mathrm{d}a-c\,\mathrm{d}b)-p[(rs-A^1_1-1)/(1+r)+(s/q)\mathcal{D}^{-1}bc]d\,\mathrm{d}c \\
&\qquad -[(r^2s-rA^1_1+1)/(1+r)-(s/r)(r+p\mathcal{D}^{-1}bc)]c\,\mathrm{d}d \\
\mathrm{d}d\, a &= -[(r(r^2s+A^1_1-r)+(rs-A^1_1+r))/(r+1)-sp\mathcal{D}^{-1}bc]d\,\mathrm{d}a \\
&\qquad +[(r^2s+A^1_1-r)/(1+r)-(s/q)\mathcal{D}^{-1}bc](qc\,\mathrm{d}b+pb\,\mathrm{d}c) \\
&\qquad +[1-s(r-(1/q)\mathcal{D}^{-1}bc)]a\,\mathrm{d}d \\
\mathrm{d}d\, b &= s(p/q)^2\mathcal{D}^{-1}bd(qd\,\mathrm{d}a-b\,\mathrm{d}c)-[(rs-A^1_1+r)/(1+r)+(s/q)\mathcal{D}^{-1}bc]d\,\mathrm{d}b \\
&\qquad -(1/q)[(r^2s-rA^1_1-r)/(1+r)-(s/r)(r+p\mathcal{D}^{-1}bc)]b\,\mathrm{d}d \\
\mathrm{d}d c &= s(q/p)^2\mathcal{D}^{-1}cd(pd\,\mathrm{d}a-c\,\mathrm{d}b)-[(rs-A^1_1+r)/(1+r)+(s/q)\mathcal{D}^{-1}bc]d\,\mathrm{d}c \\
&\qquad -(1/p)[(r^2s-rA^1_1-r)/(1+r)-(s/r)(r+p\mathcal{D}^{-1}bc)]c\,\mathrm{d}d \\
\mathrm{d}d\, d &= s\mathcal{D}^{-1}d^2(rd\,\mathrm{d}a-qc\,\mathrm{d}b-pb\,\mathrm{d}c)+[A^1_1+s(1-r+(p/r^2)\mathcal{D}^{-1}bc)]d\,\mathrm{d}d.
\end{aligned}
\tag{2.15}
$$

These relations are new results. Note that when $s \neq 0$, they are no longer *quadratic* relations (in contrast to the commutation relations between Maurer–Cartan forms and a, b, c, d). For the special solution $s = 0, A^1_1 = r^{-1}$ of (2.14) the above relations first appeared in [22] according to our knowledge (see also [25]).

Remark 1. The commutation relations (2.1) can be expressed as

$$\hat{R}(T \otimes T) = (T \otimes T)\hat{R} \tag{2.16}$$

(or $\hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}$) where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.17}$$

and

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-p^{-1} & qp^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} = q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + qp^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} q-p^{-1} & 0 \\ 0 & q \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.18}$$

For $s = 0$ the relations (2.15) simplify drastically. In particular, they become *quadratic* relations. The corresponding two solutions of (2.14) are $A^1_1 = r^{\pm 1}$ and (2.15) can be expressed as

$$\hat{R}^{\pm 1}(T \otimes \mathrm{d}T) = (q/p)^{\pm 1}(\mathrm{d}T \otimes T)\hat{R}^{\mp 1} \tag{2.19}$$

(cf [13, 14, 20]). Application of the exterior derivative, assuming $\mathrm{d}^2 = 0$ and the graded Leibniz rule, leads in both cases to

$$\hat{R}(\mathrm{d}T \otimes \mathrm{d}T) = -(q/p)(\mathrm{d}T \otimes \mathrm{d}T)\hat{R}^{-1} \tag{2.20}$$

which determines commutation relations between the differentials:

$$\begin{aligned}
&(\mathrm{d}a)^2 = (\mathrm{d}b)^2 = (\mathrm{d}c)^2 = (\mathrm{d}d)^2 = 0 \\
&\mathrm{d}b\,\mathrm{d}a = -q\mathrm{d}a\,\mathrm{d}b \qquad \mathrm{d}c\,\mathrm{d}a = -p\mathrm{d}a\,\mathrm{d}c \\
&\mathrm{d}d\,\mathrm{d}a = -\mathrm{d}a\,\mathrm{d}d \qquad \mathrm{d}c\,\mathrm{d}b = -pq^{-1}\mathrm{d}b\,\mathrm{d}c - (p-q^{-1})\mathrm{d}a\,\mathrm{d}d \\
&\mathrm{d}d\,\mathrm{d}b = -p\mathrm{d}b\,\mathrm{d}d \qquad \mathrm{d}d\,\mathrm{d}c = -q\mathrm{d}c\,\mathrm{d}d.
\end{aligned}$$

The last set of relations can also be found in [29]. For the $s = 0, A^1_1 = r$ calculus the commutation relations and their formulation in terms of the $\hat{R}$ matrix appeared in [13]†

Is it possible to express the commutation relations (2.22) or (2.15) of the general bicovariant differential calculus on $\mathrm{GL}_{p,q}(2)$ (only) in terms of T, the $\hat{R}$-matrix and the parameter s? In [28] it was conjectured that the family of differential calculi on $\mathrm{GL}_{p,q}(2)$ given in theorem 2.1 can be described by

$$\mathrm{d}T_1\,T_2 = \hat{R}_{12}T_1\,\mathrm{d}T_2\hat{R}_{12} + t(\hat{R}_{12} + q^{-1})T_1\,\mathrm{d}T_2(\hat{R}_{12} + q^{-1}) \tag{2.21}$$

with a parameter t, an expression which appeared in [14]. This is not true, however, since the latter are *quadratic* relations, whereas (2.15) involve *quartic* powers of algebra elements when $s \neq 0$.

† There is a misprint in the third equation of (22) in [13].

Remark 2. In [23] (see also [22, 24, 14]) differential calculi on $\mathrm{GL}_{p,q}(2)$ have been constructed from differential calculi on a quantum plane with generating 'coordinates' x^i, $i = 1, 2$. It is assumed that the mapping

$$\delta(x) = T\dot{\otimes}x \qquad \delta(\mathrm{d}x) = \mathrm{d}T\dot{\otimes}x + T\dot{\otimes}\,\mathrm{d}x$$

extends to a morphism of differential algebras. Starting with two specific differential algebras on the quantum plane† one recovers, in particular, the two bicovariant differential calculi with $s = 0$. There are four additional differential calculi which—by comparison of the corresponding relations in [23] with (2.15)—are not bicovariant.

Bicovariant first-order differential calculi always admit an extension to higher orders [1]. Differential forms of higher order are obtained by applying d to 1-forms (and then also higher forms) using $\mathrm{d}^2 = 0$ and the graded Leibniz rule. Bicovariance guarantees that there are commutation relations between the 1-forms which are compatible with these structures. In the following sections we will need some more results from [27].

Theorem 2.2. Let $r \neq 0, -1, \pm i$. With the exception‡ of the two solutions $s = 1/(1+r), A^1_1 = -1/(1+r)$ and $s = 1/[2(1+r)], A^1_1 = r/[2(1+r)]$ of (2.14) the commutation relations of the Maurer–Cartan forms for the bicovariant differential calculi in theorem 2.1 are

$$\begin{aligned}
(\theta^1)^2 &= \frac{1-r(1+rN)}{r+r^2(1+N)}\theta^2\theta^3\\
(\theta^2)^2 &= (\theta^3)^2 = 0\\
(\theta^4)^2 &= \frac{r(1-r+N)}{1+r(1+N)}\theta^2\theta^3\\
\theta^2\theta^1 &= \frac{1}{1+r(1+N)}\left(-r[2+N(1+r-s(1+r^2))]\theta^1\theta^2\right.\\
&\qquad\left.+[1-r-rN(r-s(1+r^2))]\theta^2\theta^4\right)\\
\theta^3\theta^1 &= \frac{1}{M}\left(-[2+N(1+r-s(1+r^2))]\theta^1\theta^3 + [r-1+rN(r-s(1+r^2))]\theta^3\theta^4\right)\\
\theta^4\theta^1 &= -\theta^1\theta^4 + (r-1)\frac{2+(r+1)N}{1+r(1+N)}\theta^2\theta^3\\
\theta^3\theta^2 &= -\theta^2\theta^3\\
\theta^4\theta^2 &= \frac{1}{1+r(1+N)}\left(-[2+N(1+r-s(1+r^2))]\theta^2\theta^4\right.\\
&\qquad\left.+[r-1-N(1-s(1+r^2))]\theta^1\theta^2\right)\\
\theta^4\theta^3 &= -\frac{1}{M}\left([r-1-N(1-s(1+r^2))]\theta^1\theta^3 + r[2+N(1+r-s(1+r^2))]\theta^3\theta^4\right)
\end{aligned} \tag{2.22}$$

† These are skew products (see [23]) of the two algebras generated, respectively, by x^i with $x^1x^2 = qx^2x^1$ and $\mathrm{d}x^i$ with $(\mathrm{d}x^i)^2 = 0$, $\mathrm{d}x^1\,\mathrm{d}x^2 = -(1/p)\mathrm{d}x^2\,\mathrm{d}x^1$.

‡ The two exceptional solutions have to be discussed separately. The corresponding analysis is simple and will not be presented in this work.

where

$$M = 1 + r + N[r^2 + r + 1 - s(r^3 + r^2 + r + 1)] \tag{2.23}$$

$$N = \frac{1}{r(r+1)}[(r^2+1)(A^1_1 - rs) - 2r]. \tag{2.24}$$

Proof. See [27]. The formulae only make sense when $M \neq 0$ and $1 + r(1+N) \neq 0$. These conditions are only violated for the two solutions of (2.14) excluded in the theorem or if $r \in \{0, -1, \pm i\}$. □

Theorem 2.3. Let $r \neq 0, \pm 1$. Then

$$\mathrm{d}f = \frac{1}{N}[\vartheta, f] \qquad \forall f \in \mathcal{A} \tag{2.25}$$

for the bicovariant calculi on $\mathrm{GL}_{p,q}(2)$. Here N is given by (2.24) and

$$\vartheta := \theta^1 + \frac{1}{r}\theta^4 \tag{2.26}$$

is a bi-coinvariant 1-form.

Proof. See [27]. Using (2.14) one finds that $N \neq 0$ iff $r \neq \pm 1$. □

More generally, (2.25) holds with f replaced by any k-form if the commutator is replaced by a graded commutator [1]. A formula of this kind plays a central role in [26] where bicovariant differential calculi are defined on more general classes of algebras.

3. Consistency of the differential algebra

The relations (2.22) show that the 'ordered' monomials $\theta^K\theta^L$ with $K < L$ form a basis of the space of 2-forms. What about the spaces of higher monomials? Here one meets the problem that the relations (2.22) do not allow a 'mechanical' ordering of higher than quadratic monomials if $s \neq 0$. For example, if we try to express the cubic monomial $\theta^2\theta^1\theta^3$ as a linear combination of ordered monomials $\theta^K\theta^L\theta^M$ with $K < L < M$ we run into an ordering cycle:

$$\theta^2\theta^1\theta^3 \to \theta^1\theta^2\theta^3 + \theta^2\theta^4\theta^3 \to \theta^1\theta^2\theta^3 + \theta^2\theta^3\theta^4 + \theta^2\theta^1\theta^3 \to \dots \tag{3.1}$$

(where we have omitted the numerical factors). In [13] this has been put forward as an argument to single out the two $s = 0$ calculi (for which ordering cycles do not appear)†. The appearance of ordering cycles does *not* mean that the ordering relations (2.22) are inconsistent. But, in principle, such ordering cycles can give rise to additional constraints or missing relations between higher monomials and thus change the apparent dimension of the space of monomials. Taking care of the numerical factors in (3.1) we can *solve* the resulting equation for the monomial $\theta^2\theta^1\theta^3$. The result is

$$\theta^2\theta^1\theta^3 = E^{-1}\left(-M\theta^1\theta^2\theta^3 + [r - 1 + rN(r - s(1+r^2))]\theta^2\theta^3\theta^4\right) \tag{3.2}$$

which shows that we can express the monomial in terms of ordered monomials if

$$E := 2 + N[1 + r - s(1 + r^2)] \tag{3.3}$$

does not vanish. $E = 0$ means $s = 1/(1+r)$, $A^1_1 = -1/(1+r)$. But this is one of the 'forbidden' solutions of (2.14) which we encountered in section 2. In this way we arrive at the following result.

† This argument would forbid the bicovariant differential calculus on $\mathrm{SL}_q(2)$ as considered, for example, in [19], cf also section 4.

Theorem 3.1. Let $r \neq 0, -1, \pm i$. With the exclusion of the solutions

(a)	$s = 1/(1+r)$,	$A^1_1 = -1/(1+r)$
(b)	$s = 1/[2(1+r)]$,	$A^1_1 = r/[2(1+r)]$
(c)	$s = -r/(1+r^3)$,	$A^1_1 = -r/[(1+r)(1-r+r^2)]$
(d)	$s = r/[(1+r)(1+r+r^2)]$,	$A^1_1 = r(1+2r)/[(1+r)(1+r+r^2)]$

of (2.14), the following holds for all the bicovariant differential calculi on $\mathrm{GL}_{p,q}(2)$ determined by theorem 2.1. Each cubic or quartic monomial in the θ^K can be expressed in a unique way as a linear combination of the ordered monomials. The latter are linearly independent. All quintic and higher monomials vanish identically.

Proof. First one has to consider all cubic monomials in the θ^K which are not already in the normal order. If an ordering cycle appears there will be a condition for the solvability of the respective equation for the monomial which we started with. All conditions additional to those already stated in theorem 2.2 are obtained from the relations considered in the following.

An attempt to order the monomial $\theta^2\theta^1\theta^2$ using (2.22) leads to†

$$F\theta^2\theta^1\theta^2 = 0 \tag{3.4}$$

with

$$F := (1+r^2)\{N^2sr[1+r-s(1+r^2)] + N[1+r-s(1-r)^2] + 2\}. \tag{3.5}$$

If $F = 0$, the monomial $\theta^2\theta^1\theta^2$ will be independent from the ordered monomials‡. We have $F = 0$ if $r = \pm i$. For $r \neq \pm i$ we can convert the condition $F = 0$ into an equation linear in A^1_1 using (2.14). Solving for A^1_1 and inserting the result into (2.14) leads to

$$(1+r)^2[s(1+r^3)+r][s(1+r)-1]^3 = 0. \tag{3.6}$$

For $r \neq -1$ there are thus two solutions, $s = -r/(1+r^3)$ and $s = 1/(1+r)$. Evaluation of A^1_1 then leads to

$$A^1_1 = -r/[(1+r)(1-r+r^2)] \qquad \text{and} \qquad A^1_1 = -1/(1+r)$$

respectively. The second solution was already excluded in theorem 2.2.

Treating the monomial $\theta^3\theta^1\theta^3$ in the same way and using $M \neq 0$ (cf the proof of theorem 2.2), we are led to consider the equation

$$(1+r)^2[s(1+r)(1+r+r^2)-r][s(1+r)-1]^3 = 0 \tag{3.7}$$

and thus to exclude the further solution

$$s = r/[(1+r)(1+r+r^2)] \qquad A^1_1 = r(1+2r)/[(1+r)(1+r+r^2)]$$

of (2.14).

In order to guarantee that $\theta^4\theta^3\theta^4$ vanishes, we have to forbid the solutions of

$$r(1+r)^2[s(1+r)-1][2s(1+r)-1]^3 = 0. \tag{3.8}$$

† Here we use $1 + r(1+N) \neq 0$, see the proof of theorem 2.2.

‡ We may consistently set it to zero, however.

But these are already excluded by the conditions of theorem 2.2.

There are no further conditions from the ordering of the remaining cubic and quartic monomials (the latter are all proportional to $\theta^1\theta^2\theta^3\theta^4$). All quintic monomials are easily shown to vanish.

To normal order a monomial like $\theta^4\theta^3\theta^2$ we may either start with $\theta^4\theta^3$ or with $\theta^3\theta^2$. One has to check that different ways of ordering such a monomial (and, more generally, a monomial in θ^K and elements of $\mathcal{A}$) lead to the same result. A general argument [30] says that it is sufficient to check this for the cubic monomials. This has been done and we have not found any additional constraint for the general bicovariant differential algebra†. It guarantees the uniqueness of the ordering procedure‡. □

4. Bicovariant differential calculi on $\mathrm{SL}_q(2)$

When $p = q$ the 'quantum determinant' $\mathcal{D}$ introduced in section 2 becomes central, i.e. it commutes with all elements of the Hopf algebra $\mathcal{A}$. The restriction

$$\mathcal{D} = ad - qbc = \mathbb{1} \tag{4.1}$$

then defines the quantum subgroup $\mathrm{SL}_q(2)$.

In the following two subsections we discuss two different ways to obtain bicovariant differential calculi on $\mathrm{SL}_q(2)$ from the general bicovariant calculus on $\mathrm{GL}_q(2)$.

Wherever we refer in this section to equations of section 2 the restriction $p = q$ (which implies $r = q^2$) is tacitly assumed.

4.1. $\mathcal{D} = \mathbb{1}$ as a constraint on the general bicovariant differential calculus on $GL_q(2)$

Imposing the condition (4.1) on a differential algebra for $\mathrm{GL}_q(2)$ requires that all 1-forms commute with $\mathcal{D}$. This means that

$$AD - qBC = \mathbf{1} \tag{4.2}$$

where $\mathbf{1}$ is the 4×4 unit matrix. For $p = q$ we have

$$AD - qBC = V\mathbf{1} \tag{4.3}$$

with

$$V := (q\beta)^2 \tag{4.4}$$

with β defined in (2.13). The constraint (4.1) becomes $V = 1$. This is satisfied when $q \in \{\pm 1, \pm i\}$. Otherwise (assuming $q \neq 0$ and using (2.14)) the parameter s is restricted to the values

$$s_\pm = \frac{1}{1 \pm q + q^2} \tag{4.5}$$

† Details of these calculations will be reported in the diploma thesis of C Reuten.

‡ This has been questioned in [28] for the calculus with $s \neq 0$.

where $q \neq -(1 \pm i\sqrt{3})/2$ for s_+ and $q \neq (1 \pm i\sqrt{3})/2$ for s_-. Then

$$A^1_1 = \begin{cases} -\dfrac{q^4 - q^3 + q^2 + 1}{q(q^2 - q + 1)} & \text{for } s_- \\ \dfrac{q^4 + q^3 + q^2 + 1}{q(q^2 + q + 1)} & \text{for } s_+. \end{cases} \tag{4.6}$$

For the general bicovariant calculus on $\mathrm{GL}_q(2)$ we obtain

$$\mathrm{d}\mathcal{D} = -\frac{J}{q^2+1}\vartheta \tag{4.7}$$

where ϑ is the bi-coinvariant 1-form defined in (2.26) and

$$J := q^2 A^1_1 + (q^6 + q^4 + 2q^2 + 1)s - q^4 - q^2 - 1. \tag{4.8}$$

Differentiation of (4.1) then leads to the constraint $\mathrm{d}\mathcal{D} = 0$ which is identically satisfied when $s = s_\pm$ since then $J = 0$.

Hence, there are indeed two bicovariant differential calculi on $\mathrm{GL}_q(2)$ which are consistent with the constraint (4.1):

(a) For $s = s_-$ the matrices A, B, C, D are given by

$$\begin{aligned}
A &= \begin{pmatrix} -\dfrac{q^4 - q^3 + q^2 + 1}{q(q^2 - q + 1)} & 0 & 0 & \dfrac{1}{q^2 - q + 1} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\dfrac{q^3 - q^2 + 1}{q(q^2 - q + 1)} & 0 & 0 & \dfrac{1-q}{q^2 - q + 1} \end{pmatrix} \\
B &= \begin{pmatrix} 0 & -(q+1) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{q-1}{q^2 - q + 1} & 0 & 0 & \dfrac{q(1-q)}{q^2 - q + 1} \\ 0 & -\left(1 + \dfrac{1}{q}\right) & 0 & 0 \end{pmatrix} \\
C &= \begin{pmatrix} 0 & 0 & -(q+1) & 0 \\ \dfrac{q-1}{q^2 - q + 1} & 0 & 0 & \dfrac{q(1-q)}{q^2 - q + 1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\left(1 + \dfrac{1}{q}\right) & 0 \end{pmatrix} \\
D &= \begin{pmatrix} \dfrac{q(q-1)}{q^2 - q + 1} & 0 & 0 & -\dfrac{q^3 - q + 1}{q^2 - q + 1} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \dfrac{q^2}{q^2 - q + 1} & 0 & 0 & -\dfrac{q^4 + q^2 - q + 1}{q(q^2 - q + 1)} \end{pmatrix}.
\end{aligned} \tag{4.9}$$

(b) For $s = s_+$ the matrices A, B, C, D take the following form:

$$
\begin{aligned}
A &= \begin{pmatrix} \frac{q^4+q^3+q^2+1}{q(q^2+q+1)} & 0 & 0 & \frac{1}{q^2+q+1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{q^3+q^2-1}{q(q^2+q+1)} & 0 & 0 & \frac{q+1}{q^2+q+1} \end{pmatrix} \\
B &= \begin{pmatrix} 0 & q-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{q+1}{q^2+q+1} & 0 & 0 & \frac{q(q+1)}{q^2+q+1} \\ 0 & \frac{1}{q}-1 & 0 & 0 \end{pmatrix} \\
C &= \begin{pmatrix} 0 & 0 & q-1 & 0 \\ \frac{q+1}{q^2+q+1} & 0 & 0 & \frac{q(q+1)}{q^2+q+1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q}-1 & 0 \end{pmatrix} \\
D &= \begin{pmatrix} \frac{q(q+1)}{q^2+q+1} & 0 & 0 & \frac{q^3-q-1}{q^2+q+1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{q^2}{q^2+q+1} & 0 & 0 & \frac{q^4+q^2+q+1}{q(q^2+q+1)} \end{pmatrix}.
\end{aligned}
\tag{4.10}
$$

The classical limit of these calculi will be discussed in section 4.3.

Theorem 4.1. Let $q \neq 0, \pm 1, \pm i$. The $s_\pm$ calculi (with $q \neq -(1 \pm i\sqrt{3})/2$, respectively $q \neq (1 \pm i\sqrt{3})/2$) are the only bicovariant differential calculi on $\mathrm{SL}_q(2)$† .

Proof. The analysis in [27] which led to our theorem 2.1 can be repeated for $\mathrm{SL}_q(2)$ (instead of $\mathrm{GL}_{p,q}(2)$). One obtains the same results as for $\mathrm{GL}_q(2)$ and simply has to add the constraint (4.1). We have just shown that this singles out the $s_\pm$ calculi. □

Remark. Let us introduce a new basis of 1-forms‡

$$\omega^K = Q^K{}_L \theta^L \tag{4.11}$$

with the matrix

$$Q := \begin{pmatrix} \frac{q^2}{q^3-1} & 0 & 0 & \frac{q^3}{q^3-1} \\ 0 & -q^2 & 0 & 0 \\ 0 & 0 & -q^2 & 0 \\ \frac{q^5}{q^3-1} & 0 & 0 & \frac{q^2(1+q-q^3)}{q^3-1} \end{pmatrix}. \tag{4.12}$$

† The $s_\pm$ calculi are also defined for $q = \pm 1$, i.e. $r = 1$. One has to check, however, whether there are additional calculi in this case (cf theorem 2.1).

‡ Cf equation (5.26) in [19].

From the commutation relations of θ^K with a, b, c, d one obtains the following commutation relations†:

$$\begin{aligned}
&\omega^1 a = q^{-1} a\omega^1 \qquad \omega^1 b = qb\omega^1 \\
&\omega^2 a = (1-q^2)b\omega^1 + a\omega^2 \qquad \omega^2 b = b\omega^2 \\
&\omega^3 a = a\omega^3 \qquad \omega^3 b = (1-q^2)a\omega^1 + b\omega^3 \\
&\omega^4 a = q^{-1}(q^2-1)^2 a\omega^1 + q^{-1}(1-q^2)b\omega^3 + qa\omega^4 \qquad \omega^4 b = q^{-1}(1-q^2)a\omega^2 + q^{-1}b\omega^4.
\end{aligned} \tag{4.13}$$

The missing relations are obtained from those listed above by replacing a and b by c and d, respectively. These commutation relations are (up to a slight change of notation) precisely those listed in [19] (cf equations (5.19) and (5.20) therein). Let us sketch how they were derived (cf also [16–18, 5]). In terms of the matrices [15]

$$L^+ = \begin{pmatrix} q^{-H/2} & (q-q^{-1})X_+ \\ 0 & q^{H/2} \end{pmatrix} \qquad L^- = \begin{pmatrix} q^{H/2} & 0 \\ -(q-q^{-1})X_- & -q^{-H/2} \end{pmatrix} \tag{4.14}$$

the (Drinfeld–Jimbo) commutation relations

$$[H, X_\pm] = \pm 2X_\pm \qquad [X_+, X_-] = \frac{q^H - q^{-H}}{q-q^{-1}} \tag{4.15}$$

of the quantum universal enveloping algebra $s\ell_q(2)$ (the 'quantum Lie algebra' of $\mathrm{SL}_q(2)$) can be written as

$$R_{12}L_2^+L_1^+ = L_1^+L_2^+R_{12} \qquad R_{12}L_2^-L_1^- = L_1^-L_2^-R_{12} \qquad R_{12}L_2^+L_1^- = L_1^-L_2^+R_{12} \tag{4.16}$$

if $q \neq 1$. The matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{4.17}$$

is related to $\hat{R}$ by multiplication with the permutation matrix (acting on a two-fold tensor product). The exterior derivative is now expressed as

$$\mathrm{d} = \mathrm{Tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}\begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix}\begin{pmatrix} \chi_1 & \chi_2 \\ \chi_3 & \chi_4 \end{pmatrix}\right] \tag{4.18}$$

where the matrix χ of operators χ_K is defined by

$$L^+S(L^-) = \mathbf{1} - (q-q^{-1})\chi \tag{4.19}$$

and S is the antipode of $sl_q(2)$. The relations (4.13) and the commutation relations which the 1-forms ω^K satisfy can now be derived from (2.1), the above equations, and the properties of the exterior derivative (see [19] for details)‡.

† Q becomes singular in the classical limit $q \to 1$. Whereas the limit $q \to 1$ exists for the θ^K, this is not so for ω^1 and ω^4.

‡ See also [2] for a construction of bicovariant differential calculi on quantum groups starting with the 'quantum Lie algebra'.

4.2. *A bicovariant differential algebra for $SL_q(2)$ as a subalgebra of a bicovariant differential algebra for $GL_q(2)$*

There is another simple way to obtain a differential calculus on $\mathrm{SL}_q(2)$ from a calculus on $\mathrm{GL}_q(2)$ (cf [20] where the special differential calculus with $s = 0$ and $A^1_1 = q^{-2}$ is considered). Let T denote the matrix with entries a, b, c, d satisfying the $\mathrm{GL}_q(2)$ commutation relations. Furthermore, let us assume that $\mathcal{D}^{-1/2}$ exists and commutes with all elements of $\mathrm{GL}_q(2)$ (note that $\mathcal{D}$ is central). Because of (4.3) it has to satisfy

$$\theta^K \mathcal{D}^{-1/2} = \pm V^{-1/2} \mathcal{D}^{-1/2} \theta^K \tag{4.20}$$

where V is defined in (4.4). For this to make sense we must have $V \neq 0$ which means $q \neq 0, \pm i$ and we have to exclude the solution $s = 1/(1+q^2)$, $A^1_1 = -1/(1+q^2)$ of (2.14). The entries of

$$\hat{T} := \mathcal{D}^{-1/2} T =: \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} \tag{4.21}$$

satisfy the $\mathrm{GL}_q(2)$ commutation relations and furthermore $\hat{\mathcal{D}} = \det_q \hat{T} = \mathbb{1}$. They generate $\mathrm{SL}_q(2)$ as a subalgebra of $\mathrm{GL}_q(2)$ and the differential calculus can be restricted to it. We can introduce corresponding Maurer–Cartan 1-forms

$$\begin{aligned}
\begin{pmatrix} \hat{\theta}^1 & \hat{\theta}^2 \\ \hat{\theta}^3 & \hat{\theta}^4 \end{pmatrix} &:= S(\hat{T}) d\hat{T} \\
&= S(T)dT + \frac{1}{N} S(T) \mathcal{D}^{1/2} [\vartheta, \mathcal{D}^{-1/2}] T \\
&= \pm V^{-1/2} S(T) dT + \frac{1}{N} (\pm V^{-1/2} - 1) \vartheta \mathbf{1} \\
&= \pm V^{-1/2} \begin{pmatrix} \theta^1 & \theta^2 \\ \theta^3 & \theta^4 \end{pmatrix} + \frac{1}{N} (\pm V^{-1/2} - 1) \begin{pmatrix} \vartheta & 0 \\ 0 & \vartheta \end{pmatrix}
\end{aligned} \tag{4.22}$$

where N is defined in (2.24) and ϑ in (2.26). To derive the last expression we made use of (2.25) and (4.20). It allows us to calculate commutation relations between the 1-forms $\hat{\theta}^K$ and the entries of T from the corresponding commutation relations of a bicovariant differential calculus on $\mathrm{GL}_q(2)$. From (4.22) we obtain

$$\hat{\theta}^K = \pm V^{-1/2} P^K{}_L \theta^L \tag{4.23}$$

with the matrix

$$P := \begin{pmatrix} 1 + (1 \mp V^{1/2})/N & 0 & 0 & (1 \mp V^{1/2})/(q^2 N) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (1 \mp V^{1/2})/N & 0 & 0 & 1 + (1 \mp V^{1/2})/(q^2 N) \end{pmatrix}. \tag{4.24}$$

If we write the commutation relations between the 1-forms $\hat{\theta}^K$ and elements of $\mathrm{GL}_q(2)$ in the form

$$\hat{\theta}^K f = \hat{\Theta}(f)^K_L \hat{\theta}^L \tag{4.25}$$

then we obtain

$$\hat{\Theta}(f) = P\Theta(f)P^{-1}. \tag{4.26}$$

The homomorphism property of Θ thus goes over to $\hat{\Theta}$, i.e.

$$\hat{\Theta}(fh) = \hat{\Theta}(f)\hat{\Theta}(h) \tag{4.27}$$

and this holds in particular if f and h are restricted to be elements of $\mathrm{SL}_q(2)$. Since

$$\mathrm{d}\begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} \begin{pmatrix} \hat{\theta}^1 & \hat{\theta}^2 \\ \hat{\theta}^3 & \hat{\theta}^4 \end{pmatrix} \tag{4.28}$$

$$\mathrm{d}\begin{pmatrix} \hat{\theta}^1 & \hat{\theta}^2 \\ \hat{\theta}^3 & \hat{\theta}^4 \end{pmatrix} = -\begin{pmatrix} \hat{\theta}^1 & \hat{\theta}^2 \\ \hat{\theta}^3 & \hat{\theta}^4 \end{pmatrix} \begin{pmatrix} \hat{\theta}^1 & \hat{\theta}^2 \\ \hat{\theta}^3 & \hat{\theta}^4 \end{pmatrix} \tag{4.29}$$

the differential algebra on $\mathrm{GL}_q(2)$ can be consistently restricted to the $\mathrm{SL}_q(2)$ subalgebra. In conclusion, each bicovariant differential calculus on $\mathrm{GL}_q(2)$ induces a corresponding bicovariant differential calculus on the subalgebra $\mathrm{SL}_q(2)$. There is no restriction of the parameter s except for the one mentioned after (4.20). A straightforward calculation now leads to the following result.

Theorem 4.2. Let $q \neq 0, \pm 1, \pm i$. In terms of the ($\mathrm{SL}_q(2)$ Maurer–Cartan) 1-forms (4.22) and the algebra elements $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \mathcal{D}^{1/2}$ *all* bicovariant differential calculi on $\mathrm{GL}_q(2)$ with the exception of the one specified by $s = 1/(1+q^2)$, $A^1_1 = -1/(1+q^2)$ are determined by

$$\hat{\theta}^K \hat{a} = (\hat{a}A^K_L + \hat{b}C^K_L)\hat{\theta}^L \qquad \hat{\theta}^K \hat{b} = (\hat{a}B^K_L + \hat{b}D^K_L)\hat{\theta}^L$$

$$\hat{\theta}^K \hat{c} = (\hat{c}A^K_L + \hat{d}C^K_L)\hat{\theta}^L \qquad \hat{\theta}^K \hat{d} = (\hat{c}B^K_L + \hat{d}D^K_L)\hat{\theta}^L$$

$$\hat{\theta}^K \mathcal{D}^{1/2} = \pm V^{1/2}\mathcal{D}^{1/2}\hat{\theta}^K.$$

For the plus sign in the last equation the matrices A, B, C, D are given by (4.10) and we have to require $q \neq -(1 \pm \mathrm{i}\sqrt{3})/2$. In case of the minus sign they are given by (4.9) and we have to assume $q \neq (1 \pm \mathrm{i}\sqrt{3})/2$.† □

It is remarkable that the parameter s does not explicitly appear in the first four relations of the theorem, but only in the last relation‡. In this way accordance is achieved with theorem 4.1.

Remark. As a consequence of theorem 4.2, when $p = q$ the parameter s can be eliminated from the commutation relations of the 'vector fields' ∇_K (generating the 'quantized Lie algebra') given in [27]. They are defined by

$$\mathrm{d}f = (\nabla_K f)\theta^K \qquad \forall f \in \mathcal{A}. \tag{4.30}$$

† Our assumptions ensure that $V \neq 0$ and $N \neq 0$ so that (4.22) is well defined. Furthermore, P^{-1} exists. $q \neq \pm 1$ is only assumed to ensure the '*all*', cf theorem 2.1.

‡ It appears, however, implicitly through the relation between the $\mathrm{GL}_q(2)$ and the $\mathrm{SL}_q(2)$ Maurer–Cartan forms. Note that $\hat{\theta}^K = \theta^K$ for $s = s_+$.

Instead we may consider the vector fields $\hat{\nabla}_K$ given by

$$\mathrm{d}f = (\hat{\nabla}_K f)\hat{\theta}^K \qquad \forall f \in \mathcal{A}. \tag{4.31}$$

Their commutation relations are obtained from this equation by applying d, using (4.29) and the commutation relations which the $\hat{\theta}^K$ have to obey. Since the latter do not depend (explicitly) on s, the same must hold for the $\hat{\nabla}_K\hat{\nabla}_L$ relations. Also the coproduct of $\hat{\nabla}_K$ (cf [27]) only depends on q. Hence we are in accordance with a result of Drinfel'd [31] (see also [32]) which says, roughly speaking, that simple Lie algebras have at most one 'quantum deformation' (as Hopf algebras). The results in [27] for $p \neq q$ show that such a statement is *not* true for the non-simple Lie algebra $g\ell(2)$†. □

4.3. The classical limit of the s_+ calculus on $SL_q(2)$

It seems that the two differential calculi with $s = s_\pm$ do not yield the ordinary differential calculus in the classical limit since $s_\pm$ does not vanish when $q \to 1$. In case of the s_- calculus we obtain in particular $\theta^K a = -a\theta^K$ for $K = 2, 3$ when $q = 1$. This indeed rules out an ordinary classical limit. Let us now consider the s_+ calculus. For $q = 1$ we obtain

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \end{pmatrix} a = a \begin{pmatrix} \theta^1 + \frac{1}{3}\vartheta \\ \theta^2 \\ \theta^3 \\ \theta^4 - \frac{1}{3}\vartheta \end{pmatrix} + b \begin{pmatrix} 0 \\ \frac{2}{3}\vartheta \\ 0 \\ 0 \end{pmatrix} \tag{4.32}$$

and

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \end{pmatrix} b = b \begin{pmatrix} \theta^1 - \frac{1}{3}\vartheta \\ \theta^2 \\ \theta^3 \\ \theta^4 + \frac{1}{3}\vartheta \end{pmatrix} + a \begin{pmatrix} 0 \\ 0 \\ \frac{2}{3}\vartheta \\ 0 \end{pmatrix} \tag{4.33}$$

where now $\vartheta = \theta^1 + \theta^4$. The Maurer–Cartan 1-forms defined in terms of the ordinary differential calculus on the classical Lie group $\mathrm{SL}(2, \mathbb{C})$ satisfy $\theta^4 = -\theta^1$. With this *additional* constraint one finds that a, b, c, d commute with the θ^K. In this way we recover the classical case. It has to be stressed, however, that the $q \to 1$ limit of the s_+ calculus is *not* the ordinary differential calculus.

Remark. Is there a consistent constraint on the 1-forms θ^K for $q \neq 1$ which reduces to $\theta^4 = -\theta^1$ in the limit $q \to 1$? Writing θ^4 as a linear combination of θ^1, θ^2 and θ^3, consistency with the commutation relations (2.9) restricts it to

$$\theta^4 = -q^{-1}\theta^1. \tag{4.34}$$

Except for special values of q this is, however, not consistent with the bicovariant differential algebra. For example, as a consequence of the last condition the expression

$$(\theta^4)^2 - q^{-2}(\theta^1)^2 = -q^{-3}(q-1)^2(q^2+q+1)\theta^2\theta^3 \tag{4.35}$$

would have to vanish.

† A counterexample also appeared in [33]. The authors constructed a special differential calculus (which is not bicovariant) on $\mathrm{GL}_{p,q}(2)$ and the corresponding Hopf algebra generated by the vector fields. In terms of suitable functions of the vector fields, the commutation relations were shown to depend only on a single parameter, but the coproduct formulae depend on two parameters (see also [27], section 5).

In the following we will explore the s_+ calculus on SL(2, $\mathbb{C}$) (i.e. with $q = 1$) in more detail. Let $x^1 := a, x^2 := b, x^3 := c, x^4 := d$. From the above commutation relations between Maurer–Cartan forms and a, b, c, d one derives

$$[\mathrm{d}x^\mu, x^\nu] = g^{\mu\nu}\tau \tag{4.36}$$

where

$$\tau := \tfrac{1}{3}\vartheta = \tau_\mu \, \mathrm{d}x^\mu \qquad (\tau_\mu) = \tfrac{1}{3}(x^4, -x^3, -x^2, x^1) \tag{4.37}$$

and

$$g^{\mu\nu} = x^\mu x^\nu + 4\left(\delta_2^{(\mu}\delta_3^{\nu)} - \delta_1^{(\mu}\delta_4^{\nu)}\right) \tag{4.38}$$

(indices in brackets are symmetrized). This matrix is degenerate, i.e. $\det(g^{\mu\nu}) = 0$, and satisfies $g^{\mu\nu}\tau_\nu = 0$. Using the relations of the s_+ calculus, one finds that τ satisfies $\mathrm{d}\tau = 0$ and $\tau^2 = 0$. Furthermore, it commutes with x^μ and anticommutes with $\mathrm{d}x^\mu$ ($\mu = 1, \ldots, 4$).

One of the 'coordinates' x^μ is redundant because of the constraint $\mathcal{D} = 1$. Let us consider the subalgebra generated by only three of them, say x^i where $i = 1, 2, 3$. Then

$$[\mathrm{d}x^i, x^j] = g^{ij}\tau \tag{4.39}$$

with

$$g^{ij} = x^i x^j + 4\delta_2^{(i}\delta_3^{j)}. \tag{4.40}$$

Since $\det(g^{ij}) = -4(x^1)^2 = -4a^2$, this is a non-degenerate symmetric matrix (a 'metric') if $a \neq 0$. The latter is just the condition allowing us to solve the determinant constraint $ad - bc = 1$ for $d = x^4$. In (4.39) the 1-form τ is independent from the 1-forms $\mathrm{d}x^i$. Indeed, an attempt to express τ as a linear combination of the differentials $\mathrm{d}x^i$ using $x^4 = (1 + x^2x^3)/x^1$ fails. The resulting equation is identically satisfied and does not tell us anything about τ.

4.4. Bicovariant differential calculus on $SL_q(2, \mathcal{R})$

The 'reality conditions'

$$a^* = a \qquad b^* = b \qquad c^* = c \qquad d^* = d \tag{4.41}$$

where * is an antilinear involution on $\mathcal{A}$ (which on complex numbers acts as complex conjugation denoted by a bar) is compatible with the $\mathrm{GL}_{p,q}(2)$ commutation relations only when $|p| = |q| = 1$. These conditions define the quantum group $\mathrm{GL}_{p,q}(2, \mathcal{R})$. The quantum group $\mathrm{SL}_q(2, \mathcal{R})$ is then obtained from $\mathrm{SL}_q(2)$ restricting q by $|q| = 1$, i.e.

$$\bar{q} = q^{-1}. \tag{4.42}$$

Assuming that

$$(f\,\mathrm{d}h)^* = \mathrm{d}(h^*)f^* \qquad \forall f, h \in \mathcal{A} \tag{4.43}$$

one finds that the s_+ calculus on $\mathrm{SL}_q(2)$ is compatible with the reality conditions. In order to verify this it is helpful to first derive the relations

$$(\theta^1)^* = -\theta^4 \qquad (\theta^2)^* = q\theta^2 \qquad (\theta^3)^* = q^{-1}\theta^3. \tag{4.44}$$

Then one has to apply * to the commutation relations between θ^K and a, b, c, d, and use (4.41) and (4.44) to show that the resulting equations are consequences of these commutation relations.

Let us turn to the classical limit ($q = 1$) of the s_+ calculus and use the notation of section 4.3. We still have (4.39) with (4.40), but now the functions x^i are *real* and g is a real metric on SL(2, $\mathcal{R}$).

Theorem 4.3. g is the maximally symmetric Lorentzian metric on $\mathrm{SL}(2, \mathcal{R})$ with negative constant curvature.

Proof. g^{ij} depends continously on x^2 and x^3. $\det(g^{ij}) = -4(x^1)^2$ is non-vanishing where $x^1 \neq 0$ and independent of x^2 and x^3. Hence, the signature of g^{ij} does not depend on x^2 and x^3. For $x^2 = x^3 = 0$ the eigenvalues of g^{ij} are $-2, +2$ and $(x^1)^2$. This proves that g^{ij} has Lorentzian signature $(-, +, +)$.

Calculating the curvature tensor of g we find

$$R_{ijk\ell} = -(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) \tag{4.45}$$

which shows that we have a space of constant curvature. Now the statement of the theorem follows since the last equation uniquely determines a metric with given signature (see, for example, [34], section 13.2). □

Differential calculi of the form (4.39) on a commutative algebra of real functions (as in the case under consideration where $q = 1$) were studied in [35].

5. Bicovariant differential calculi on $\mathrm{SU}_q(2)$

Analogues of unitary matrices are defined by

$$S(T) = T^\dagger := \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \tag{5.1}$$

where * is an involution of the algebra $\mathrm{GL}_{p,q}(2)$. This means that we have to identify

$$b = -q\mathcal{D}c^* \qquad d = \mathcal{D}a^*. \tag{5.2}$$

Consistency with the commutation relations of $\mathrm{GL}_{p,q}(2)$ enforces

$$p = \bar{q} \tag{5.3}$$

(where the bar denotes complex conjugation) and

$$\begin{aligned} &ac = qca \qquad ac^* = \bar{q}c^*a \qquad cc^* = c^*c \qquad a^*a = \mathbb{1} - c^*c \qquad aa^* = \mathbb{1} - q\bar{q}c^*c \\ &\mathcal{D}\mathcal{D}^* = \mathbb{1} \qquad a\mathcal{D} = \mathcal{D}a \qquad a^*\mathcal{D} = \mathcal{D}a^* \qquad c\mathcal{D} = (\bar{q}/q)\mathcal{D}c \qquad c^*\mathcal{D} = (q/\bar{q})\mathcal{D}c^*. \end{aligned} \tag{5.4}$$

These relations define the quantum group $\mathrm{U}_{\bar{q},q}(2)$ as a quantum subgroup of $\mathrm{GL}_{p,q}(2)$. The quantum subgroup $\mathrm{SU}_q(2)$ [36] is then obtained by imposing the further constraint $\mathcal{D} = \mathbb{1}$ which requires $\bar{q} = q$, i.e. q has to be real. Two bicovariant differential calculi on $\mathrm{SU}_q(2)$ were found by Woronowicz who denoted them as the $4D_\pm$ calculi [1]. It was recently demonstrated that these are the only bicovariant differential calculi on $\mathrm{SU}_q(2)$ [10]. We have shown (using different techniques) that also on $\mathrm{SL}_q(2)$ there are only two bicovariant differential calculi. The $4D_\pm$ calculi on $\mathrm{SU}_q(2)$ are obtained from our $s_\pm$ calculi by imposing the condition (5.1)†.

† In [10] the commutation relations between 1-forms and algebra elements are expressed in terms of a set of 1-forms Ω_K which are related to our s_+ Maurer–Cartan forms θ^K by $\Omega_1 = \theta^3$, $\Omega_2 = -q\theta^2$, $\Omega_3 = -(q/\sqrt{1+q^2})(\theta^1 - \theta^4)$ and $\Omega_4 = \mp q^2/(1+q^2)\vartheta$.

6. Conclusions

The main result of section 2 are the commutation relations (2.15) between the generators a, b, c, d and their differentials for the most general bicovariant differential calculus on $\mathrm{GL}_{p,q}(2)$. In terms of the Maurer–Cartan forms, the corresponding relations are much simpler and were derived in [27]. The derivation of (2.15) became necessary for comparison with the results of Manin and in order to settle a question raised in [28] concerning an R-matrix formulation for the bicovariant calculi found in [27] (see the remarks in section 2). We have checked these relations carefully using the computer algebra software REDUCE [37].

An apparent consistency problem for the calculi with $s \neq 0$ mentioned in [13] is resolved in section 3. An attempt was made in [13] to single out the two $s = 0$ calculi which admit a simple R-matrix formulation. This would rule out the $4D_{\pm}$ calculi on $\mathrm{SU}_q(2)$ since we have shown that they correspond to the $s_{\pm}$ calculi on $\mathrm{GL}_q(2)$ (see section 5) and this is precisely the calculus considered in [19].

The authors of [20] mentioned that the seemingly obvious way to carry differential calculus from $\mathrm{GL}_q(2)$ over to $\mathrm{SL}_q(2)$ by imposing the determinant constraint $\mathcal{D} = 1$ does *not* work. We have shown, however, that it does work if the right choice of bicovariant *differential calculus on* $\mathrm{GL}_q(2)$ *is made. It works for our* $s_{\pm}$ *calculi.* An alternative, but less direct way towards differential calculus on $\mathrm{SL}_q(2)$ starting with the $s = 0, A_1^1 = q^{-2}$ calculus on $\mathrm{GL}_q(2)$ was followed in [20]. We have exploited this procedure in section 4.2 for the most general bicovariant differential calculus on $\mathrm{GL}_q(2)$. This led to our theorem 4.2 which shows that the complicated structure of differential calculi on $\mathrm{GL}_{p,q}(2)$ greatly simplifies when $p = q$.

In conclusion, there are two different ways to obtain bicovariant differential calculi on $\mathrm{SL}_q(2)$ *from those on* $\mathrm{GL}_q(2)$*:*

(a) We can start from the $\mathrm{GL}_q(2)$ differential calculi with $s = s_{\pm}$ and impose the determinant constraint on the differential algebra (see section 4.1).

(b) We can start from *any* of the $\mathrm{GL}_q(2)$ differential calculi and restrict it to the $\mathrm{SL}_q(2)$ subalgebra (as discussed in section 4.2).

Both procedures lead to the same result in accordance with our uniqueness theorem 4.1.

Among the two bicovariant differential calculi on $\mathrm{SL}_q(2)$ only the s_+ calculus has a reasonable classical limit, although even in this case we do not obtain the ordinary differential calculus as $q \to 1$. We have shown in sections 4.3 and 4.4 that the resulting differential calculus for $q = 1$ is of the form of the deformed differential calculus (on a commutative function algebra) discussed in [35]. It remains to be seen to what extent the physical ideas in [35] apply to the classical limit of bicovariant differential calculi on quantum groups and beyond.

The fact that the space of 1-forms is four-dimensional for the quantum groups $\mathrm{SL}_q(2)$ and $\mathrm{SU}_q(2)$ whereas it is three-dimensional for the ordinary differential calculus on SL(2) and SU(2) is usually regarded as an unpleasant feature of bicovariant differential calculus†. The above-mentioned relation with the work in [35] may turn this apparently negative aspect into a positive one, however.

† A differential calculus with a three-dimensional space of 1-forms on $\mathrm{SU}_q(2)$ has been considered in [36]. It is not known whether this calculus can be characterized in a natural way in order to distinguish it from the many other consistent (and not necessarily bicovariant) calculi.

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